

On the Fourth Power Moment of Fourier Coefficients of Cusp Form

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Abstract: Let $a(n)$ be the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \geq 12$ for the full modular group and $A(x) = \sum_{n \leq x} a(n)$. In this paper, we establish an asymptotic formula of the fourth power moment of $A(x)$ and prove that

$$\int_1^T A^4(x) dx = \frac{3}{64\kappa\pi^4} s_{4;2}(\tilde{a}) T^{2\kappa} + O(T^{2\kappa-\delta_4+\varepsilon})$$

with $\delta_4 = 1/8$, which improves the previous result.

Keywords: Cusp form; Fourier coefficient; mean value; asymptotic formula

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1 Introduction and main result

Let $a(n)$ be the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \geq 12$ for the full modular group. In 1974, Deligne [2] proved the following profound result

$$a(n) \ll n^{(\kappa-1)/2} d(n), \quad (1.1)$$

where $d(n)$ denotes the Dirichlet divisor function and the implied constant in \ll is absolute. Suppose $x \geq 2$ and define

$$A(x) := \sum_{n \leq x} a(n). \quad (1.2)$$

It is well known that $A(x)$ has no main term and $A(x) \ll x^{\kappa/2-1/6+\varepsilon}$. In 1973, Joris [5] proved that

$$A(x) = \Omega_{\pm}(x^{\kappa/2-1/4} \log \log \log x).$$

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In 1990, Ivić [3] showed that there exist two points t_1 and t_2 in the interval $[T, T + CT^{1/2}]$ such that

$$A(t_1) > Bt_1^{\kappa/2-1/4}, \quad A(t_2) < -Bt_2^{\kappa/2-1/4},$$

where $B > 0$, $C > 0$ are constants. It is conjectured that

$$A(x) \ll x^{(\kappa-1)/2+1/4+\varepsilon}$$

is true for every ε . The evidence in support of this conjecture has been given by Ivić [3], who proved the following square mean value formula of $A(x)$, i.e.

$$\int_1^T A^2(x) dx = C_2 T^{\kappa+1/2} + B(T),$$

where

$$C_2 = \frac{1}{(4\kappa+2)\pi^2} \sum_{n=1}^{\infty} a^2(n) n^{-\kappa-1/2},$$

$$B(T) \ll T^{\kappa} \log^5 T, \quad B(T) = \Omega\left(T^{\kappa-1/4} \frac{(\log \log \log T)^3}{\log T}\right).$$

In [3], Ivić also proved the upper bound of eighth power moment of $A(x)$, that is

$$\int_1^T A^8(x) dx \ll T^{4\kappa-1+\varepsilon}.$$

Cai [1] studied the third and fourth power moments of $A(x)$. He proved that

$$\int_1^T A^3(x) dx = C_3 T^{(6\kappa+1)/4} + O(T^{(6\kappa+1)/4-\delta_3+\varepsilon}), \quad (1.3)$$

$$\int_1^T A^4(x) dx = C_4 T^{2\kappa} + O(T^{2\kappa-\delta_4+\varepsilon}), \quad (1.4)$$

where $\delta_3 = 1/14$, $\delta_4 = 1/23$ and

$$C_3 := \frac{3}{4(6\kappa+1)\pi^3} \sum_{\substack{n,m,k \in \mathbb{N} \\ \sqrt{n}+\sqrt{m}=\sqrt{k}}} (nmk)^{-\kappa/2-1/4} a(n)a(m)a(k),$$

$$C_4 := \frac{3}{64\kappa\pi^4} \sum_{\substack{n,m,k,\ell \in \mathbb{N} \\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{\ell}}} (nmk\ell)^{-\kappa/2-1/4} a(n)a(m)a(k)a(\ell).$$

In [10], Zhai proved that (1.3) holds for $\delta_3 = 1/4$. Following the approach of Tsang [9], Zhai [10] proved that the equation (1.4) holds for $\delta_4 = 2/41$. This approach used the method of exponential sums. In particular, if the exponent pair conjecture is true, namely, if $(\varepsilon, 1/2+\varepsilon)$ is an exponent pair, then the equation (1.4) holds for $\delta_4 = 1/14$.

Later, combining the method of [4] and a deep result of Robert and Sargos [8], Zhai [12] proved that the equation (1.4) holds for $\delta_4 = 3/28$. By a unified approach, Zhai [11] proved that the asymptotic formula

$$\int_1^T A^k(x) dx = \mathcal{C}_k T^{1+k(2\kappa-1)/4} + O(T^{1+k(2\kappa-1)/4-\delta_k+\varepsilon})$$

holds for $3 \leq k \leq 7$, where \mathcal{C}_k and $0 < \delta_k < 1$ are explicit constants.

The aim of this paper is to improve the value of $\delta_4 = 3/28$, which is achieved by Zhai [12]. The main result is the following

Theorem 1.1 *We have*

$$\int_1^T A^4(x) dx = \frac{3}{64\kappa\pi^4} s_{4;2}(\tilde{a}) T^{2\kappa} + O(T^{2\kappa-\delta_4+\varepsilon})$$

with $\delta_4 = 1/8$, where

$$s_{4;2}(\tilde{a}) = \sum_{\substack{n,m,k,\ell \in \mathbb{N}^* \\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{\ell}}} \frac{a(n)a(m)a(k)a(\ell)}{(nmk\ell)^{\kappa/2+1/4}}.$$

Notation. Throughout this paper, $a(n)$ be the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \geq 12$ for the full modular group; $d(n)$ denote the Dirichlet divisor function; $\tilde{a}(n) := a(n)n^{-\kappa/2+1/2}$; $\|x\|$ denotes the distance from x to the nearest integer, i.e., $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$. $[x]$ denotes the integer part of x ; $n \sim N$ means $N < n \leq 2N$; $n \asymp N$ means $C_1 N \leq n \leq C_2 N$ with positive constants C_1, C_2 satisfying $C_1 < C_2$. ε always denotes an arbitrary small positive constant which may not be the same at different occurrences. We shall use the estimates $d(n) \ll n^\varepsilon$. Suppose $f : \mathbb{N} \rightarrow \mathbb{R}$ is any function satisfying $f(n) \ll n^\varepsilon$, $k \geq 2$ is a fixed integer. Define

$$s_{k;\ell}(f) := \sum_{\substack{n_1, \dots, n_\ell, n_{\ell+1}, \dots, n_k \in \mathbb{N}^* \\ \sqrt{n_1} + \dots + \sqrt{n_\ell} = \sqrt{n_{\ell+1}} + \dots + \sqrt{n_k}}} \frac{f(n_1)f(n_2) \cdots f(n_k)}{(n_1 n_2 \cdots n_k)^{3/4}}, \quad 1 \leq \ell < k. \quad (1.5)$$

We shall use $s_{k;\ell}(f)$ to denote both of the series (1.5) and its value. Suppose $y > 1$ is a large parameter, and we define

$$s_{k;\ell}(f; y) := \sum_{\substack{n_1, \dots, n_\ell, n_{\ell+1}, \dots, n_k \leq y \\ \sqrt{n_1} + \dots + \sqrt{n_\ell} = \sqrt{n_{\ell+1}} + \dots + \sqrt{n_k}}} \frac{f(n_1)f(n_2) \cdots f(n_k)}{(n_1 n_2 \cdots n_k)^{3/4}}, \quad 1 \leq \ell < k.$$

2 Preliminary Lemmas

Lemma 2.1 *If $g(x)$ and $h(x)$ are continuous real-valued functions of x and $g(x)$ is monotonic, then*

$$\int_a^b g(x)h(x)dx \ll \left(\max_{a \leq x \leq b} |g(x)| \right) \left(\max_{a \leq u < v \leq b} \left| \int_u^v h(x)dx \right| \right).$$

Proof. See Tsang [9], Lemma 1. ■

Lemma 2.2 Suppose $A, B \in \mathbb{R}$, $A \neq 0$. Then we have

$$\int_T^{2T} t^\alpha \cos(A\sqrt{t} + B) dt \ll T^{1/2+\alpha} |A|^{-1}.$$

Proof. It follows from Lemma 2.1 easily. ■

Lemma 2.3 If $n, m, k, \ell \in \mathbb{N}$ such that $\sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{\ell} \neq 0$, then there hold

$$|\sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{\ell}| \gg (nmk\ell)^{-1/2} \max(n, m, k, \ell)^{-3/2},$$

respectively.

Proof. See Kong [7], Lemma 3.2.1. ■

Lemma 2.4 Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be any function satisfying $f(n) \ll n^\varepsilon$. Then we have

$$|s_{k;\ell}(f) - s_{k;\ell}(f; y)| \ll y^{-1/2+\varepsilon}, \quad 1 \leq \ell < k,$$

where $k \geq 2$ is a fixed integer.

Proof. See Zhai [11], Lemma 3.1. ■

Lemma 2.5 Suppose $1 \leq N \leq M$, $1 \leq K \leq L$, $N \leq K$, $M \asymp L$, $0 < \Delta \ll L^{1/2}$. Let $\mathcal{A}_1(N, M, K, L; \Delta)$ denote the number of solutions of the following inequality

$$0 < |\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{\ell}| < \Delta$$

with $n \sim N$, $m \sim M$, $k \sim K$, $\ell \sim L$. Then we have

$$\mathcal{A}_1(N, M, K, L; \Delta) \ll \Delta L^{1/2} NMK + NKL^{1/2+\varepsilon}.$$

Especially, if $\Delta L^{1/2} \gg 1$, then

$$\mathcal{A}_1(N, M, K, L; \Delta) \ll \Delta L^{1/2} NMK.$$

Proof. See Zhai [12], Lemma 5. ■

Lemma 2.6 Suppose $N_j \geq 2$ ($j = 1, 2, 3, 4$), $\Delta > 0$. Let $\mathcal{A}_\pm(N_1, N_2, N_3, N_4; \Delta)$ denote the number of solutions of the following inequality

$$0 < |\sqrt{n_1} + \sqrt{n_2} \pm \sqrt{n_3} - \sqrt{n_4}| < \Delta$$

with $n_j \sim N_j$ ($j = 1, 2, 3, 4$), $n_j \in \mathbb{N}^*$. Then we have

$$\mathcal{A}_\pm(N_1, N_2, N_3, N_4; \Delta) \ll \prod_{j=1}^4 (\Delta^{1/4} N_j^{7/8} + N_j^{1/2}) N_j^\varepsilon.$$

Proof. See Zhai [12], Lemma 3. ■

3 Proof of Theorem 1.1

In this section, we shall prove the theorem. We begin with the following truncated formula, which is proved by Jutila [6], i.e.,

$$A(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \leq N} \frac{a(n)}{n^{\kappa/2+1/4}} x^{\kappa/2-1/4} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{\kappa/2+\varepsilon} N^{-1/2}), \quad (3.1)$$

where $1 \leq N \ll x$.

Suppose $T \geq 10$. By a splitting argument, it is sufficient to prove the result in the interval $[T, 2T]$. Take $y = T^{3/4}$. For any $T \leq x \leq 2T$, by the truncated formula (3.1), we get

$$A(x) = \frac{1}{\sqrt{2\pi}} \mathcal{R}(x) + O(x^{\kappa/2+\varepsilon} y^{-1/2}), \quad (3.2)$$

where

$$\mathcal{R}(x) := x^{\kappa/2-1/4} \sum_{n \leq y} \frac{a(n)}{n^{\kappa/2+1/4}} \cos(4\pi\sqrt{nx} - \pi/4).$$

We have

$$\begin{aligned} \int_T^{2T} A^4(x) dx &= \frac{1}{4\pi^4} \int_T^{2T} \mathcal{R}^4(x) dx + O(T^{2\kappa+1/4+\varepsilon} y^{-1/2} + T^{2\kappa+1+\varepsilon} y^{-2}) \\ &= \frac{1}{4\pi^4} \int_T^{2T} \mathcal{R}^4(x) dx + O(T^{2\kappa-1/8+\varepsilon}). \end{aligned} \quad (3.3)$$

Let

$$g = g(n, m, k, \ell) := \begin{cases} \frac{a(n)a(m)a(k)a(\ell)}{(nmk\ell)^{\kappa/2+1/4}}, & \text{if } n, m, k, \ell \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

According to the elementary formula

$$\cos a_1 \cos a_2 \cdots \cos a_h = \frac{1}{2^{h-1}} \sum_{(i_1, i_2, \dots, i_{h-1}) \in \{0, 1\}^{h-1}} \cos(a_1 + (-1)^{i_1} a_2 + \cdots + (-1)^{i_{h-1}} a_h),$$

we can write

$$\mathcal{R}^4(x) = S_1(x) + S_2(x) + S_3(x) + S_4(x), \quad (3.4)$$

where

$$\begin{aligned} S_1(x) &:= \frac{3}{8} \sum_{\substack{n, m, k, \ell \leq y \\ \sqrt{n} + \sqrt{m} = \sqrt{k} + \sqrt{\ell}}} g x^{2\kappa-1}, \\ S_2(x) &:= \frac{3}{8} \sum_{\substack{n, m, k, \ell \leq y \\ \sqrt{n} + \sqrt{m} \neq \sqrt{k} + \sqrt{\ell}}} g x^{2\kappa-1} \cos(4\pi(\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{\ell})\sqrt{x}), \end{aligned}$$

$$\begin{aligned}
S_3(x) &:= \frac{1}{2} \sum_{\substack{n,m,k,\ell \leq y \\ \sqrt{n}+\sqrt{m}+\sqrt{k} \neq \sqrt{\ell}}} g x^{2\kappa-1} \cos\left(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}-\sqrt{\ell})\sqrt{x}-\frac{\pi}{2}\right), \\
S_4(x) &:= \frac{1}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1} \cos\left(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{\ell})\sqrt{x}-\pi\right).
\end{aligned}$$

By (1.1) and Lemma 2.4, we get

$$\begin{aligned}
\int_T^{2T} S_1(x) dx &= \frac{3}{8} s_{4,2}(a(n)n^{-\kappa/2+1/2}; y) \int_T^{2T} x^{2\kappa-1} dx \\
&= \frac{3}{8} s_{4,2}(\tilde{a}; y) \int_T^{2T} x^{2\kappa-1} dx \\
&= \frac{3}{8} s_{4,2}(\tilde{a}) \int_T^{2T} x^{2\kappa-1} dx + O(T^{2\kappa} y^{-1/2+\varepsilon}) \\
&= \frac{3}{8} s_{4,2}(\tilde{a}) \int_T^{2T} x^{2\kappa-1} dx + O(T^{2\kappa-3/8+\varepsilon}). \tag{3.5}
\end{aligned}$$

We now proceed to consider the contribution of $S_4(x)$. Applying Lemma 2.2 and (1.1), we get

$$\begin{aligned}
\int_T^{2T} S_4(x) dx &= \frac{1}{8} \sum_{n,m,k,\ell \leq y} g \int_T^{2T} x^{2\kappa-1} \cos\left(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{\ell})\sqrt{x}-\pi\right) dx \\
&\ll \sum_{n,m,k,\ell \leq y} \frac{g T^{2\kappa-1/2}}{\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{\ell}} \\
&= T^{2\kappa-1/2} \sum_{n,m,k,\ell \leq y} \frac{a(n)a(m)a(k)a(\ell)}{(nmk\ell)^{(\kappa-1)/2}(nmk\ell)^{3/4}} \cdot \frac{1}{\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{\ell}} \\
&\ll T^{2\kappa-1/2} \sum_{n,m,k,\ell \leq y} \frac{d(n)d(m)d(k)d(\ell)}{(nmk\ell)^{3/4}\ell^{1/2}} \\
&\ll T^{2\kappa-1/2+\varepsilon} \sum_{n,m,k,\ell \leq y} \frac{1}{(nmk)^{3/4}\ell^{5/4}} \\
&\ll T^{2\kappa-1/2+\varepsilon} y^{1/2} \ll T^{2\kappa-1/8+\varepsilon}. \tag{3.6}
\end{aligned}$$

Now let us consider the contribution of $S_2(x)$. By the first derivative test, we have

$$\begin{aligned}
\int_T^{2T} S_2(x) dx &\ll \sum_{\substack{n,m,k,\ell \leq y \\ \sqrt{n}+\sqrt{m} \neq \sqrt{k}+\sqrt{\ell}}} g \min\left(T^{2\kappa}, \frac{T^{2\kappa-1/2}}{|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{\ell}|}\right) \\
&\ll x^\varepsilon \mathcal{G}(N, M, K, L), \tag{3.7}
\end{aligned}$$

where

$$\mathcal{G}(N, M, K, L) = \sum_{\substack{\sqrt{n}+\sqrt{m} \neq \sqrt{k}+\sqrt{\ell} \\ n \sim N, m \sim M, k \sim K, \ell \sim L \\ 1 \leq N \leq M \leq y \\ 1 \leq K \leq L \leq y}} g \cdot \min\left(T^{2\kappa}, \frac{T^{2\kappa-1/2}}{|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{\ell}|}\right).$$

If $M \geq 200L$, then $|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{\ell}| \gg M^{1/2}$, so the trivial estimate yields

$$\mathcal{G}(N, M, K, L) \ll \frac{T^{2\kappa-1/2+\varepsilon} N M K L}{(N M K L)^{3/4} M^{1/2}} \ll T^{2\kappa-1/2+\varepsilon} y^{1/2} \ll T^{2\kappa-1/8+\varepsilon}.$$

If $L \geq 200M$, we can get the same estimate. So later we always suppose that $M \asymp L$. Let $\eta = \sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{\ell}$. Write

$$\mathcal{G}(N, M, K, L) = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3, \quad (3.8)$$

where

$$\begin{aligned} \mathcal{G}_1 &:= T^{2\kappa} \sum_{0 < |\eta| \leq T^{-1/2}} g, \\ \mathcal{G}_2 &:= T^{2\kappa-1/2} \sum_{T^{-1/2} < |\eta| \leq 1} g |\eta|^{-1}, \\ \mathcal{G}_3 &:= T^{2\kappa-1/2} \sum_{|\eta| > 1} g |\eta|^{-1}. \end{aligned}$$

We estimate \mathcal{G}_1 first. By Lemma 2.5, we get

$$\begin{aligned} \mathcal{G}_1 &\ll \frac{T^{2\kappa+\varepsilon}}{(N M K L)^{3/4}} \mathcal{A}_1(N, M, K, L; T^{-1/2}) \\ &\ll \frac{T^{2\kappa+\varepsilon}}{(N M K L)^{3/4}} (T^{-1/2} L^{1/2} N M K + N K L^{1/2}) \\ &\ll T^{2\kappa-1/2+\varepsilon} (N K)^{1/4} + T^{2\kappa+\varepsilon} (N K)^{1/4} L^{-1} \\ &\ll T^{2\kappa-1/2+\varepsilon} y^{1/2} + T^{2\kappa+\varepsilon} (N K)^{1/4} L^{-1} \\ &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} (N K)^{1/4} L^{-1}. \end{aligned} \quad (3.9)$$

On the other hand, by Lemma 2.6, without loss of generality, we assume that $N \leq K \leq L$ and obtain

$$\begin{aligned} \mathcal{G}_1 &\ll \frac{T^{2\kappa+\varepsilon}}{(N M K L)^{3/4}} \mathcal{A}_-(N, M, K, L; T^{-1/2}) \\ &\ll \frac{T^{2\kappa+\varepsilon}}{(N M K L)^{3/4}} (T^{-1/8} N^{7/8} + N^{1/2}) (T^{-1/8} K^{7/8} + K^{1/2}) (T^{-1/4} L^{7/4} + L) \\ &\ll T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-1/8} N^{3/8} + 1) (T^{-1/8} K^{3/8} + 1) (T^{-1/4} L^{3/4} + 1) \\ &\ll T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-1/4} (N K)^{3/8} + T^{-1/8} K^{3/8} + 1) (T^{-1/4} L^{3/4} + 1) \\ &\ll T^{2\kappa-1/4+\varepsilon} (N K)^{1/8} L^{-1/2} \\ &\quad + T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-1/8} K^{3/8} + 1) (T^{-1/4} L^{3/4} + 1) \\ &\ll T^{2\kappa-1/4+\varepsilon} L^{-1/4} + T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-3/8} L^{9/8} + 1) \\ &\ll T^{2\kappa-1/4+\varepsilon} + T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-3/8} L^{9/8} + 1). \end{aligned} \quad (3.10)$$

From (3.9) and (3.10), we get

$$\mathcal{G}_1 \ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} \cdot \min\left(\frac{(NK)^{1/4}}{L}, \frac{T^{-3/8}L^{9/8}+1}{(NK)^{1/4}L^{1/2}}\right).$$

Case 1 If $L \gg T^{1/3}$, then $T^{-3/8}L^{9/8} \gg 1$, we get

$$\begin{aligned} \mathcal{G}_1 &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} \cdot \min\left(\frac{(NK)^{1/4}}{L}, \frac{T^{-3/8}L^{9/8}}{(NK)^{1/4}L^{1/2}}\right) \\ &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} \left(\frac{(NK)^{1/4}}{L}\right)^{1/2} \left(\frac{T^{-3/8}L^{9/8}}{(NK)^{1/4}L^{1/2}}\right)^{1/2} \\ &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-3/16+\varepsilon} L^{-3/16} \ll T^{2\kappa-1/8+\varepsilon}. \end{aligned} \quad (3.11)$$

Case 2 If $L \ll T^{1/3}$, then $T^{-3/8}L^{9/8} \ll 1$. By noting that $M \asymp L \asymp \max(N, M, K, L)$ and Lemma 2.3, we have

$$T^{-1/2} \gg |\eta| \gg (nmk\ell)^{-1/2} \max(n, m, k, \ell)^{-3/2} \asymp (NK)^{-1/2} L^{-5/2}.$$

Hence, we obtain

$$\begin{aligned} \mathcal{G}_1 &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} \min\left(\frac{(NK)^{1/4}}{L}, \frac{1}{(NK)^{1/4}L^{1/2}}\right) \\ &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} \left(\frac{(NK)^{1/4}}{L}\right)^{1/4} \left(\frac{1}{(NK)^{1/4}L^{1/2}}\right)^{3/4} \\ &= T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} (NK)^{-1/8} L^{-5/8} \\ &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} (T^{-1/2})^{1/4} \ll T^{2\kappa-1/8+\varepsilon}. \end{aligned} \quad (3.12)$$

Combining (3.11) and (3.12), we get

$$\mathcal{G}_1 \ll T^{2\kappa-1/8+\varepsilon}. \quad (3.13)$$

Now, we estimate \mathcal{G}_2 . By a splitting argument, we get that there exists some δ satisfying $T^{-1/2} \ll \delta \ll 1$ such that

$$\mathcal{G}_2 \ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4}\delta} \times \sum_{\substack{\delta < |\eta| \leq 2\delta \\ \eta \neq 0}} 1.$$

By Lemma 2.5, we get

$$\begin{aligned} \mathcal{G}_2 &\ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4}\delta} \mathcal{A}_1(N, M, K, L; 2\delta) \\ &\ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4}\delta} (\delta L^{1/2} NMK + NKL^{1/2}) \\ &= T^{2\kappa-1/2+\varepsilon} (NK)^{1/4} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} (NK)^{1/4} L^{-1} \\ &\ll T^{2\kappa-1/2+\varepsilon} y^{1/2} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} (NK)^{1/4} L^{-1} \\ &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} (NK)^{1/4} L^{-1}. \end{aligned} \quad (3.14)$$

On the other hand, by Lemma 2.6, without loss of generality, we assume that $N \leq K \leq L$ and obtain

$$\begin{aligned}
\mathcal{G}_2 &\ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4}\delta} \times \mathcal{A}_-(N, M, K, L; 2\delta) \\
&\ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4}\delta} (\delta^{1/4}N^{7/8} + N^{1/2})(\delta^{1/4}K^{7/8} + K^{1/2})(\delta^{1/2}L^{7/4} + L) \\
&\ll T^{2\kappa-1/2+\varepsilon}(NK)^{-1/4}L^{-1/2}\delta^{-1}(\delta^{1/4}N^{3/8} + 1)(\delta^{1/4}K^{3/8} + 1)(\delta^{1/2}L^{3/4} + 1) \\
&\ll T^{2\kappa-1/2+\varepsilon}(NK)^{-1/4}L^{-1/2}\delta^{-1}(\delta^{1/2}(NK)^{3/8} + \delta^{1/4}K^{3/8} + 1)(\delta^{1/2}L^{3/4} + 1) \\
&\ll T^{2\kappa-1/2+\varepsilon}(NK)^{1/8}L^{-1/2}\delta^{-1/2} \\
&\quad + T^{2\kappa-1/2+\varepsilon}(NK)^{-1/4}L^{-1/2}\delta^{-1}(\delta^{1/4}K^{3/8} + 1)(\delta^{1/2}L^{3/4} + 1) \\
&\ll T^{2\kappa-1/4+\varepsilon}L^{-1/4} + T^{2\kappa-1/2+\varepsilon}(NK)^{-1/4}L^{-1/2}\delta^{-1}(\delta^{3/4}L^{9/8} + 1). \tag{3.15}
\end{aligned}$$

From (3.14) and (3.15), we get

$$\mathcal{G}_2 \ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}\delta^{-1} \cdot \min\left(\frac{(NK)^{1/4}}{L}, \frac{\delta^{3/4}L^{9/8} + 1}{(NK)^{1/4}L^{1/2}}\right).$$

Case 1 If $\delta \gg L^{-3/2}$, then $\delta^{3/4}L^{9/8} \gg 1$, we get (recall $\delta \gg T^{-1/2}$)

$$\begin{aligned}
\mathcal{G}_2 &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}\delta^{-1} \cdot \min\left(\frac{(NK)^{1/4}}{L}, \frac{\delta^{3/4}L^{9/8}}{(NK)^{1/4}L^{1/2}}\right) \\
&\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}\delta^{-1} \left(\frac{(NK)^{1/4}}{L}\right)^{1/2} \left(\frac{\delta^{3/4}L^{9/8}}{(NK)^{1/4}L^{1/2}}\right)^{1/2} \\
&\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}\delta^{-5/8}L^{-3/16} \\
&\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}T^{5/16}L^{-3/16} \ll T^{2\kappa-1/8+\varepsilon}. \tag{3.16}
\end{aligned}$$

Case 2 If $\delta \ll L^{-3/2}$, then $\delta^{3/4}L^{9/8} \ll 1$. By Lemma 2.3, we have

$$\delta \gg |\eta| \gg (nmk\ell)^{-1/2} \max(n, m, k, \ell)^{-3/2} \asymp (NK)^{-1/2}L^{-5/2}.$$

Therefore, we obtain (recall $\delta \gg T^{-1/2}$)

$$\begin{aligned}
\mathcal{G}_2 &\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}\delta^{-1} \cdot \min\left(\frac{(NK)^{1/4}}{L}, \frac{1}{(NK)^{1/4}L^{1/2}}\right) \\
&\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}\delta^{-1} \left(\frac{(NK)^{1/4}}{L}\right)^{1/4} \left(\frac{1}{(NK)^{1/4}L^{1/2}}\right)^{3/4} \\
&\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}\delta^{-1}(NK)^{-1/8}L^{-5/8} \\
&\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}\delta^{-1}\delta^{1/4} \\
&\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon}T^{3/8} \ll T^{2\kappa-1/8+\varepsilon}. \tag{3.17}
\end{aligned}$$

Combining (3.16) and (3.17), we get

$$\mathcal{G}_2 \ll T^{2\kappa-1/8+\varepsilon}. \tag{3.18}$$

For \mathcal{G}_3 , by a splitting argument and Lemma 2.5 again, we get

$$\begin{aligned}
\mathcal{G}_3 &\ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4}\delta} \times \sum_{\substack{\delta < |\eta| \leq 2\delta \\ \delta \gg 1}} 1 \\
&\ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4}\delta} \cdot \delta L^{1/2} NMK \ll T^{2\kappa-1/2+\varepsilon} (NK)^{1/4} \\
&\ll T^{2\kappa-1/2+\varepsilon} y^{1/2} \ll T^{2\kappa-1/8+\varepsilon}.
\end{aligned} \tag{3.19}$$

Comnining (3.7), (3.8), (3.13), (3.18) and (3.19), we get

$$\int_T^{2T} S_2(x) dx \ll T^{2\kappa-1/8+\varepsilon}. \tag{3.20}$$

In the same way, we can prove that

$$\int_T^{2T} S_3(x) dx \ll T^{2\kappa-1/8+\varepsilon}. \tag{3.21}$$

From (3.3)-(3.6), (3.20) and (3.21), we get

$$\int_T^{2T} A^4(x) dx = \frac{3}{32\pi^4} s_{4;2}(\tilde{a}) \int_T^{2T} x^{2\kappa-1} dx + O(T^{2\kappa-1/8+\varepsilon}), \tag{3.22}$$

which implies Theorem 1.1 immediately.

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